An Elementary Expression of Density Function of Normal Laplace Distribution

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Abstract- *Properties such as infinite divisibility, skew, excess kurtosis and suitable model the heavy-tailed distribution characteristic of real finance asset yield make the Normal-Laplace distribution a good candidate model in option pricing. However, a few constraints to Normal-Laplace parameter and density function lack of the enclosed expression may lead to the traditional moment estimation and maximum likelihood estimation failure, therefore, we put forward elementary enclosed expression of density function of Normal-Laplace distribution under some special conditions, try to remedy above lack.*

Keyword- *Normal-Laplace distribution; density function; enclosed expression*

1. INTRODUCTION

The Normal-Laplace distribution (NL) is first introduced by Reed and Jorgensen (2004). Meanwhile, they put forward the probability density function of the NL distribution, the cumulative distribution function and moment generating function (MGF). They also briefly introduced the advantages of NL distribution in modeling a number of areas of size distribution, including economic (income and profit distribution), financial (stock price), geography (human settlements population distribution), physical (particle size model distribution) and geology (oil) field analysis. Although they focused on the introduction and presentation of NL distribution, however, they did not sufficiently discuss its applications in detail in the above areas. Reed (2007) used the NL distribution in option pricing analysis, and introduced the generalized normal Laplace distribution (GNL). Properties such as infinite divisibility, skew, excess kurtosis and can better model the heavy-tailed distribution characteristic of real finance asset yield make the GNL distribution a good candidate model in option pricing. Simos et al (2010) consider that the generalized normal-Laplace distribution is a useful law for modelling asymmetric data exhibiting excess kurtosis. They constructed goodness-of-fit tests for this distribution which utilize the corresponding moment generating function, and its empirical counterpart. BIN Tong (2013) further extended the application of NL distribution in option price. He made the distribution as a condition of heavy-tailed distribution and analyzed a bivariate GARCH option price model. [Normal distribution]

CEHN Xiao-Hong (2006) proved the probability distribution of enterprise-scale obeys normal distribution, and the probability distribution of growth rate obeys Laplace distribution, which is a natural phenomenon, and not affected by his own attributes and external environmental factors. Due to the asymmetric Laplace distribution has an explicit expression, it facilitate the calculation of the digital features and parameter estimation. Hence for stock index futures investors, using asymmetric Laplace distribution to calculate the VAR and CVAR of the stock index futures yield would be a better choice. ZENG Wu-Yi(2012) used asymmetric Laplace distribution to fit Shanghai and Shenzhen stock daily ,weekly yield date. His results showed that asymmetric Laplace distribution can reflect the spike, thick tail, skew features of Shanghai and Shenzhen stock daily, weekly yield date well than normal distribution.

In terms of the model parameter estimation, since Reed (2007) put constraints to GNL parameter and density function lack of the enclosed expression may lead to the traditional moment estimation and maximum likelihood estimation failure. Therefore, Lonica Groparu-cojocaru and Louis G.Doray (2013) used quadratic distance estimator (QDE) method in the GNL model parameter estimation.

The analysis of Tobit model with non-normal error distribution is extended to the case of asymmetric Laplace distribution (ALD). Since the ALD probability density function is known to be continuous but not differentiable, the usual mode-finding algorithms such as maximum likelihood can be difficult and result in the inconsistent parameter estimates. Nuttanan Wichitaksorn (2013) use a survey dataset on the wage earnings of Thai male workers and compare the Tobit model with normal and ALD errors through the model marginal likelihood, his results reveal that the model with the ALD error is preferred. Alessandro Barbiero (2014) proposed an alternative discrete skew Laplace distribution by using the general approach of discretizing a continuous distribution while retaining its survival function. He explored distribution's properties and compared it to a Laplace distribution on integers recently proposed in the literature.

Katherine and Paul (2013) introduced a dimension reduction method for model-based clustering via a finite mixture of shifted asymmetric Laplace distributions. It is based on existing work within the Gaussian paradigm and relies on identification of a reduced subspace. Their clustering approach is illustrated on simulated and real data, where it performs favorably compared to its Gaussian analogue.

Reed and Jorgensen (2004) firstly introduced the normal-Laplace distribution, then, K.K. Jose et al (2008) discussed a first order autoregressive process with normal-Laplace stationary marginal distribution and various properties. The process gives a combination of Gaussian as well as non-Gaussian time series models for the first time and is free from the zero defect problem. They also discussed the applications in modelling data from various contexts. Recently, Shams Harandi (2013) put forward a new class of skew distributions containing both unimodal and bimodal distributions with flexible hazard rate behavior. He investigated some distributional properties of this class, and presented a characterization, a generating method and parameter estimation. Finally, he also examined the model using real data sets. However, a few constraints to Normal-Laplace parameter and density function lack of the enclosed expression may lead to the traditional moment estimation and maximum likelihood estimation failure, In this paper, we put forward elementary enclosed expression of density function of Normal-Laplace distribution under some special cases, try to remedy above lack.

2. ESTIMATION

Reed and Jorgensen (2004) first introduce a new distribution, the NL distribution. Then Reed (2006) showed its usefulness in modeling the option price. If Φ and ϕ is respectively the cumulative distribution function (cdf) and the probability density function (pdf) of a standard normal distribution. And the Mills ratio is

$$R(x)=\frac{1-\Phi(x)}{\phi(x)} \tag{1}$$

Then the pdf of NL can be expressed as

$$F(x)=\Phi\left(\frac{x-\nu}{\tau}\right)-\phi\left(\frac{x-\nu}{\tau}\right)\frac{\beta R\left(\alpha\tau-\frac{x-\nu}{\tau}\right)-\alpha R\left(\beta\tau+\frac{x-\nu}{\tau}\right)}{\alpha+\beta} \tag{2}$$

We shall refer to this as the Normal-Laplace distribution and write $X\sim NL(\alpha,\beta,\nu,\ \tau^2)$ to indicate that X follows this distribution. $\nu\in R$ stands for the location parameter, $\tau>0$ is the scale parameter, $\alpha>0$ and $\beta>0$ determine the degree of obesity of the right and left end of the tail. And, the greater the values of α and β are, the thinner their tails will be. In particular, when $\alpha\to\infty$, $\beta\to\infty$, NL distribution would degenerate into a normal distribution. The pdf of NL distribution is

$$f(x)=\frac{\alpha\beta}{\alpha+\beta}\phi\left(\frac{x-\nu}{\tau}\right)\left(R\left(\alpha\tau-\frac{x-\nu}{\tau}\right)+R\left(\beta\tau+\frac{x-\nu}{\tau}\right)\right) \tag{3}$$

It is well known that the moment generating function (MGF) of the X is the product of the MGFs s of its normal and Laplace components. Precisely, we can express X=W+Z, where Z follows a $N(\nu,\tau^2)$ distribution and W follows a skewed Laplace distribution (Kotzetal. 2001) with probability density function (pdf)

$$f_w(x)=\begin{cases}\frac{\alpha\beta}{\alpha+\beta}e^{\beta x},x\ge 0,\\ \frac{\alpha\beta}{\alpha+\beta}e^{-\alpha x},x>0,\end{cases} \tag{4}$$

Notice that

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt \tag{5}$$

We can rewritten the Mills ratio as follows:

$$R(x)=e^{\frac{x^2}{2}}\left[\frac{1}{2}-\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt\right] \tag{6}$$

Denote region

$S_x=\{(s,t)|0\le s\le x,\ 0\le t\le x\}$,

$D_x=\{(s,t)|\ s^2+t^2\le x^2,\ s\ge 0,t\ge 0\}$.

It is obviously that for arbitrary x>0, we have relationship:

$$D_x\subset S_x\subset D_{\sqrt{2}x}.$$

In the light of properties of two dimension integral, the following integral inequality is true:

$$\iint_{D_x}e^{-\frac{s^2+t^2}{2}}dsdt\le\iint_{S_x}e^{-\frac{s^2+t^2}{2}}dsdt\le\iint_{D_{\sqrt{2}x}}e^{-\frac{s^2+t^2}{2}}dsdt \tag{7}$$

After simple calculate, we obtain

$$\iint_{S_x}e^{-\frac{s^2+t^2}{2}}dsdt=\int_0^x e^{-\frac{s^2}{2}}ds\int_0^x e^{-\frac{t^2}{2}}dt=\left(\int_0^x e^{-\frac{t^2}{2}}dt\right)^2;$$

$$\iint_{D_x}e^{-\frac{s^2+t^2}{2}}dsdt=\int_0^{\frac{\pi}{2}}d\theta\int_0^x e^{-\frac{r^2}{2}}rdr=\frac{\pi}{2}(1-e^{-\frac{x^2}{2}});$$

$$\iint_{D_{\sqrt{2}x}}e^{-\frac{s^2+t^2}{2}}dsdt=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sqrt{2}x}e^{-\frac{r^2}{2}}rdr=\frac{\pi}{2}(1-e^{-x^2})$$

Therefore, inequality (7) is equivalent to the following one:

$$\sqrt{\frac{\pi}{2}\left(1-e^{-\frac{x^2}{2}}\right)}\le\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt\le\sqrt{\frac{\pi}{2}\left(1-e^{-x^2}\right)} \tag{8}$$

According to the intermediate value theorem, there exists a constant k which is only relate to x, such that

$$\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt=\sqrt{\frac{\pi}{2}\left(1-e^{-kx^2}\right)} \tag{9}$$

In this way, we get an elementary expression for the Mills ratio

$$R(x) = e^{\frac{x^2}{2}}\left[\frac{1}{2} - \sqrt{\frac{\pi}{2}\left(1 - e^{-kx^2}\right)}\right] \quad (10)$$

Bring function $R(x)$(10) into (3), then we obtain an element representation for density function of Normal-Laplace distribution:

$$f(x) = \frac{\alpha\beta}{\alpha+\beta}\cdot\frac{e^{-\frac{1}{2}\left(\frac{x-\nu}{\tau}\right)^2}}{\sqrt{2\pi}}\cdot\left\{e^{\frac{1}{2}\left(\alpha\tau-\frac{x-\nu}{\tau}\right)^2}\left[\frac{1}{2} - \sqrt{\frac{\pi}{2}\left(1-e^{-k_1\left(\alpha\tau-\frac{x-\nu}{\tau}\right)^2}\right)}\right]\right.$$
$$\left. + e^{\frac{1}{2}\left(\beta\tau+\frac{x-\nu}{\tau}\right)^2}\left[\frac{1}{2} - \sqrt{\frac{\pi}{2}\left(1-e^{-k_2\left(\beta\tau+\frac{x-\nu}{\tau}\right)^2}\right)}\right]\right\} \quad (11)$$

Where constant k_1 is only relate to $\alpha\tau - \frac{x-\nu}{\tau}$, constant k_2 is only relate to $\beta\tau + \frac{x-\nu}{\tau}$, and $\frac{1}{2} < k_1 < 1$, $\frac{1}{2} < k_2 < 1$.

3. CONCLUSIONS

It is obviously that the right side of representation (11) is a simple compound of elementary functions. In some special regions in the parameters space, we have a few simple elementary expressions for density function of NL distribution.

Corollary 1 If $\beta\tau + \frac{x-\nu}{\tau}$ is sufficient small, then we have approximate representation

$$f(x) \approx K_1 \cdot \exp\left(\frac{1}{2}\alpha^2\tau^2 - \alpha(x-\nu)\right).$$

Where

$$K_1 = \frac{\alpha\beta}{\alpha+\beta}\cdot\left[\frac{1}{\sqrt{2\pi}} - \sqrt{1-\exp[-k_1(\alpha+\beta)^2\tau^2]}\right].$$

Corollary 2 If $\alpha\tau - \frac{x-\nu}{\tau}$ is sufficient small, then we have approximate expression

$$f(x) \approx K_2 \cdot \exp\left(\frac{1}{2}\beta^2\tau^2 + \beta(x-\nu)\right).$$

Where

$$K_2 = \frac{\alpha\beta}{\alpha+\beta}\cdot\left[\frac{1}{\sqrt{2\pi}} - \sqrt{1-\exp[-k_2(\alpha+\beta)^2\tau^2]}\right].$$

Corollary 3 If τ is sufficient small, then we have approximate representation

$$f(x) \approx K_3 \cdot \left[\exp(-\alpha(x-\nu)) + \exp(\beta(x-\nu))\right].$$

Where

$$K_3 = \frac{1}{2\sqrt{2\pi}}\cdot\frac{\alpha\beta}{\alpha+\beta}.$$

Corollary 4 If τ is sufficient small, and $\alpha = \beta$, then we have approximate expression

$$f(x) \approx K_4 \cdot \cosh(-\alpha(x-\nu)).$$

Where

$$K_4 = \frac{\alpha}{2\sqrt{2\pi}}.$$

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